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1993 J. Phys. A: Math. Gen. 26 6367

(http://iopscience.iop.org/0305-4470/26/22/033)

J. Phys. A: Math. Gen. 26 (1993) 6367–6377. Printed in the UK

Choices in the harmonic balance technique

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Received 10 June 1993

Abstract. We apply the harmonic balance technique in low orders to three types of oscillator model. For three equations of the van der Pol type with rational nonlinear terms, used to model electronic oscillators, we examine the choice between: (a) rationalizing before expanding in harmonics, and (b) obtaining Fourier coefficients of nonlinear terms. Alternative (b) is shown to be more accurate, in first order, for the models of Scott–Murata and of Walker and Connelly, which apply to circuits with an inverse tangent nonlinear component. For these equations, in the case of alternative (a), we find acceptable dependence of the second-order corrections on the bifurcation parameter. When the harmonic balance method is used to set up a semi-classical quantization treatment of a nonlinear conservative oscillator, values of energies are slightly improved by the use of alternative (b). For conservative oscillators with cubic or fifth power forces, we compare the standard method using the acceleration equation with an alternative using the energy equation. The first is shown to be more accurate in low order.

1. Introduction: choices in harmonic balance

The standard harmonic balance method for approximating periodic solutions of a nonlinear ordinary differential equation involves the following steps:

(I) Select a trial solution which is a truncated Fourier series, either with terms $a_n \cos(npt)$ alone, n up to N, or with both sine and cosine terms, as appropriate. Insert this solution in the equation, and ignore any higher harmonics (terms with $n>N$) generated by the nonlinear terms.

(II) Set equal to zero the coefficients of the retained harmonics, thus obtaining a set of coupled nonlinear equations for the frequency p and the amplitudes a_n in the trial solution. Solve these equations.

The method has a long history, but still finds fresh physical and engineering applications. It has recently been used to interpret the behaviour of new oscillator circuits [1], and to give a starting point for quasi-classical calculations of bound states [2].

Here we are concerned solely with the use of this method in low order, to obtain some understanding of how the frequency and amplitude of an autonomous oscillation depend on parameters. We do not look at its use at high order as a means of numerical approximation to solutions for autonomous or forced oscillations [3]. Further we are not considering the complications that arise [4, 5] when the restoring force is of mixed parity.

Even in this restricted use of the method, alternative procedures are possible. For example, one may use elliptic functions rather than cosines [6]. This is more accurate for conservative oscillators with forces expanded as powers of the displacement, which

0305-4470/93/226367+11$07.50

is not surprising when we note that the exact solution of a pure cubic equation

$$d^2x/dt^2+kx^3=0 \tag{1}$$

is an elliptic function. Rational trial solutions, for example the form $a\cos(pt)/(1+b\cos(2pt))$, are proposed in [7–9]. These have been exploited in recent work on some Hamiltonian systems in dynamical astronomy [10, 11].

We shall examine two other choices within the harmonic balance method. The first applies to equations in which the nonlinear terms contain ratios of polynomials; it is the choice between preliminary rationalization or obtaining Fourier coefficients by integration. Consider the equation of van der Pol type

$$d^2x/dt^2+x+qf(x)\,dx/dt=0 \tag{2}$$

with $f(x)$ an even function, and the lowest-order approximation

$$x=a\cos(pt).$$

Three different equations in which $f(x)$ has a denominator depending on x^2 have been proposed, by Scott [12] (also by Murata *et al* [13, 14]), by Walker and Connelly [15] and by Ceschia and Zecchin [16], as models of oscillating devices for which the van der Pol equation is inadequate. An example is the ECAM device (emitter-coupled astable multivibrator) for which the voltage can be represented as an inverse tangent of the current. We shall denote these as the Scott equation, WC equation and CZ equation for brevity. The nonlinear functions $f(x)$ for the first two can be expressed as

$$h(1-g/(hx^2+1)) \tag{3}$$

where $h=1$, $g>1$ for the Scott equation, and $h=-1$, $g<1$ for the WC equation. The singularity in the WC equation means that only the range $-1<x<1$ is considered.

Scott [12] comments that for large q the slow sections of the curve of $x(t)$, using (2), are concave towards the time axis, while practical relaxation oscillators generate waveforms in which the slow sections are exponential decays, concave outwards. His equation is designed to simulate this behaviour.

In his analysis of the WC equation, Mickens [1] applies harmonic balance in the lowest order, as well as other techniques. He first rationalizes the equation, multiplying throughout by a factor x^2-1, and then introduces the trial solution $a\cos(pt)$. Results obtained in this manner need not be identical with those obtained by performing the integral which gives the Fourier sine coefficient of the nonlinear term, with this same trial solution

$$\frac{1}{\pi}\int_0^{2\pi} f(a\cos(pt))(-pa\sin(pt)\sin(pt))\,d(pt). \tag{4}$$

We maintain that if the integral can be analytically evaluated, its use is to be preferred. It explicitly follows the description we have given of step I in the method. The strength of the harmonic balance method, at low orders, is not in giving close approximations to solutions, but rather in approximating such averaged quantities as frequency and mean amplitude. The integration performs an averaging process.

The motivation for the pre-rationalization in [1] can be found in the way Mickens [17] expresses conditions for the applicability of harmonic balance. He limits the method to differential equations in which each term is a product of odd total power, such as, for example, x^3, $x^2\,dx/dt$, $x(dx/dt)^2$ or $x^2\,d^2x/dt^2$. This is to exclude the possibility of

generating constant terms or $\cos(2pt)$ terms, which lead to difficulties [4, 5]. All this means, however, for the application of integration is that we have to verify that the unwanted Fourier components are indeed absent. We have done this in each case studied.

Mickens' rationalization procedure is an unambiguous way of performing an approximation by truncated Fourier expansion. So we have to compare the results of both methods before rejecting rationalization. As we shall show, the integration method agrees with values for the amplitude of the stable limit cycle obtained, using other techniques, by Scott [12] and Murata *et al* [14] for the Scott equation, and by Walker and Connelly [15] for their equation, but the rationalization method does not. For the WC equation the amplitude agrees better with numerical results in [1] if integration is used.

We shall also compare the two methods for a conservative oscillator example from [2]. Here the quasi-classical energies obtained by the integration method are marginally better, in comparisons with exact numerical solutions.

The second choice comes into play for a conservative system, in which the differential equation has a constant first integral—which we shall denote as energy. We can make use of the energy to sharpen the choice between two methods. If two harmonic balance methods are applied to the differential equation, at the same order of approximation, the preferred one should be the one which leaves smaller fluctuations in the energy. This argument is used in the astronomical applications [10, 11] as a reason to prefer a rational trial solution. In appendix 1 we compare standard and rational harmonic balance from this viewpoint in a much simpler conservative system, the linear plus cubic oscillator. We confirm that the rational trial solution can have this desirable effect.

However we can see no *a priori* reason to reject an alternative method, which eliminates low harmonics from the energy equation. We have examined this choice for two equations in which the standard method is well documented [6, 17]. These are the cubic equation (1) and the fifth-power equation.

$$d^2x/dt^2+kx^5=0. \tag{5}$$

We find that at first order the standard method gives appreciably better results for the cubic equation. Neither method is very good for the fifth-order equation, as compared with results using elliptic functions. Not surprisingly, the two methods converge as we go to second and third order, although more slowly for the fifth-power equation than for the cubic one.

Computer algebra packages make it easy to investigate alternative methods for a variety of oscillator equations. Most of these packages contain a facility to convert automatically from powers and products of sines and cosines to sines and cosines of sum and difference angles. We have used Maple and Derive, which have this facility, and REDUCE, where the necessary rules can be given by the user. Automatic integration is also provided, although we find some desirable integrals defeat the packages available to us.

2. Equations of the van der Pol type: rationalize or integrate?

We consider the first-order trial solutions $a\cos(pt)$, using the form (3) of the nonlinear factor in the damping term in Scott or WC equations. Preliminary multiplication

throughout by hx^2+1 gives

$$(-p^2+1)\ a\cos(pt)\ (ha^2\cos^2(pt)+1)$$
$$+hq((ha^2\cos^2(pt)+1-g)(-ap\sin(pt)))=0 \tag{6}$$

from which the coefficient of $\cos(pt)$ yields $p=1$, while the coefficient of $\sin(pt)$ yields

$$a^2=4(g-1)/h. \tag{7}$$

Now for the Scott equation we have $h=1$, $g>1$, while for the WC equation we have $h=-1$, $g>1$. So in each case we have

$$a=2\sqrt{|g-1|}. \tag{8}$$

As emphasized by Mickens [1], bifurcation theory confirms that a limit cycle exists for $g<1$ for the WC equation, and that the amplitude, for small enough $1-g$, rises as $\sqrt{(1-g)}$. But bifurcation theory cannot specify how small this has to be. (The relationship between harmonic balance and Hopf bifurcation has been investigated by Allwright [18].) When we proceed by integration, we have

$$(-p^2+1)a\cos(pt)+(hq/\pi)I\sin(pt)+\text{higher harmonics}$$

where the integral is

$$I=\int_0^{2\pi}(1-g/(ha^2\cos^2(pt)+1))\ (-ap\sin^2(pt))\ \mathrm{d}(pt). \tag{9}$$

For $h=1$, $p=1$, this is

$$I=(\pi/a)(2g\sqrt{(a^2+1)}-a^2-2g)$$

leading to the conclusion that

$$a=2\sqrt{(g(g-1))}. \tag{10}$$

For $h=-1$ the integral is

$$I=(\pi/a)((a^2-2g)-2ig\sqrt{(a^2-1)}).$$

Since the singularity in the WC equation implies we consider only $a^2<1$, this leads to

$$a=2\sqrt{(g(1-g))}. \tag{11}$$

Noting the restrictions on the g values, both the results (10) and (11) are

$$a=2\sqrt{|g(g-1)|}$$

which, for g close enough to 1, is still in agreement with bifurcation theory. These results can be compared with other analytical approximations, and with numerical simulation. For the Scott equation an averaging method [12] gives $2\sqrt{(g(g-1))}$ as the amplitude of a stable limit cycle (Appendix 2). Murata *et al* [14] present an expansion in Fourier components, of which the first term has this amplitude. Walker and Connelly [15] give the form $2\sqrt{(g(1-g))}$ for the amplitude of the stable limit cycle in their model, again from an averaging method.

Table 1. Limit cycle amplitude for the WC equation.

g	q	numerical	$2\sqrt{(1-g)}$	$2\sqrt{(g(1-g))}$
0.85	0.1	0.714	0.775	0.715
0.85	0.5	0.714	0.775	0.715
0.95	0.1	0.434	0.447	0.436
0.95	0.5	0.434	0.447	0.436

Mickens [1] presents results of numerical solution of the ordinary differential equation, for two values of g. Table 1 compares the amplitudes from his numerical solution with the two forms obtained by harmonic balance. The results support the conclusion of harmonic balance that the amplitude does not depend on q. They also support the form (11) of the amplitude rather than the form (8).

The CZ oscillator equation [16] is more elaborate, having three parameters, with

$$f(x)=\left(1+gh^2\frac{x^2-h^2}{(x^2+h^2)^2}\right) \tag{12}$$

and has been studied only for large q. Oscillations are possible for $g>1$. We can carry out the first-order harmonic balance method for (12) in either way. We can express the results in terms of $y=a^2/h^2$, since h only appears as a scaling factor. From a pre-rationalized calculation

$$y=\sqrt{(g^2+12g-4)}-g-2 \tag{13}$$

while from integration, y satisfies

$$y^3+(1-4g)y^2+4g(g-1)y=0. \tag{14}$$

From (13), and also from the root $y=4g-1-\sqrt{(8g+1)}$ of (14), the amplitude rises as $\sqrt{(g-1)}$ for g close to 1. In this case the amplitude derived from integration rises more rapidly than $\sqrt{(g-1)}$ away from $g=1$; that derived from pre-rationalization rises less rapidly.

Again the use of an averaging method gives the amplitude in agreement with that derived from integration (appendix 2). We have carried out some simulations for $q=0.2$, $h=1$, with g in the range 1.1 to 1.5. These indicate that the amplitude increases more rapidly than $\sqrt{(g-1)}$.

3. Second-order harmonic balance for equations of van der Pol type

We now examine the second-order corrections to the amplitude, for the Scott and WC equations. A remarkable feature of the lowest-order approximation to all equations of the van der Pol type (2) is that the nonlinearity parameter q is not involved. However, at second order, with the trial solution

$$x=a\cos(pt)+b\cos(3pt)+c\sin(3pt)$$

for the original van der Pol equation, one can obtain a set of results for low q:

$$a=2+q^2/64 \qquad p=1-q^2/16$$
$$b=-3q^2/32 \qquad c=-q/4.$$

The reliability of the lowest-order results implicitly depends on the possibility of such an expansion in powers of a small parameter. Mickens [17] has emphasized the need to show that second-order amplitudes are small; the corrections at second order to the first-order amplitude should also be small.

We first examine whether q is the appropriate small parameter in the Scott or WC equations. With the second-order trial solution for these equations, the terms that arise from preliminary rationalization include one which is proportional to

$$a^2c\sin(pt). \tag{15}$$

When we insert an expanded set of parameters:

$$a=2\sqrt{|g-1|}+Aq^m \qquad p=1+Pq^n$$
$$b=Bq^r \qquad c=Cq^s$$

the coefficient of $\sin(3pt)$ yields the result $s=1$. The coefficients of $\cos(3pt)$ and $\cos(pt)$ are consistent with $n=r=2$, and do not lead to any conclusion about m. As a result of using $s=1$, the term (15) expands to include a term

$$4(g-1)^2Cq.$$

This is the only term of order q. The lowest-order terms containing a factor q^m have q^{m+1}, and so we conclude that $m=0$; the correction to a is independent of q, rather than having q^2 dependence as for the van der Pol equation.

However there is another relevant small parameter here, the bifurcation parameter $1-g$. We find that

$$a=2\sqrt{|1-g|}(1+(g-1)/2g).$$

So as g approaches 1, the correction falls off more rapidly than the first-order value. The amplitude c is

$$2q|1-g|^{3/2}/(2g-1) \tag{16}$$

in this approximation. The amplitude b and the correction to p are more complicated,

$$b=\frac{q^2(1-g)^{5/2}(12g^2-10g+1)}{32(5g^2-5g+1)(2g-1)^2} \tag{17}$$

$$p=1-\frac{q^2(g-1)^2(2-3g)}{16(5g^2-5g+1)(1-2g)} \tag{18}$$

The expressions (16) to (18) all depend on (small) q in the same manner as for the van der Pol equation, and fall off more sharply than a as g approaches 1. This is satisfactory.

However the expressions (16) to (18) all diverge as g approaches $\frac{1}{2}$, in the case of the singular WC equation. Now this singular equation can only be meaningful with amplitude less than 1. With the form (8) for the first-order amplitude, derived by way of preliminary rationalization, this indicates a minimum allowable value $g=\frac{3}{4}$, not $g=\frac{1}{2}$. The divergence in B and P at $g=\frac{1}{2}+1/\sqrt{2}$ is also puzzling.

Unfortunately we cannot obtain the integrals needed for the second-order calculation by the alternative method; these integrands contain denominators such as

$$1+a\cos(pt)+b\cos(3pt)+c\sin(3pt).$$

So we cannot resolve the difficulty by using this method. Our case for integration as opposed to preliminary rationalization rests on the better approximation to the amplitude at first order.

4. Integral and pre-rationalized treatments for a conservative oscillator

As a simple illustration of the difference between these versions of the harmonic balance method, we consider the equation used by Mickens [2] in advocating the use of harmonic balance as a preliminary to a WKB treatment of a quantized conservative oscillator. This is

$$\mathrm{d}^2x/\mathrm{d}t^2+x+kx/(1+gx^2)^2=0 \tag{19}$$

corresponding to a potential

$$\frac{x^2}{2}\left(1+\frac{k}{1+gx^2}\right).$$

From the form of the potential it is clear that for small gx^2 the system becomes a harmonic oscillator with spring constant $1+k$, while for large gx^2 it becomes a harmonic oscillator with spring constant 1, with the potential augmented by a constant term $k/2g$, not affecting the frequency.

In lowest order, preliminary rationalization gives

$$p^2-1=k/(1+3a^2g/2+5a^4g^2). \tag{20}$$

With this form of trial solution the action quantization equation reduces to the form

$$pa^2=(2n+1)h. \tag{21}$$

Mickens [2] substitutes for p in (21), solves for a, and evaluates the energy by inserting this amplitude in the potential. He compares it with the energy found [19] by numerical solution of the Schrödinger equation.

The integral to obtain the $\cos(pt)$ term of the nonlinear part of (19) is $ka(a^2g+1)^{-3/2}$, giving

$$p^2=1+k(a^2g+1)^{-3/2}. \tag{22}$$

Both (20) and (22) behave as expected for a^2g small and a^2g large. At small a^2g (22) gives a rather similar form to (20)

$$p^2-1=k/(1+3a^2g/2+3a^4g^2/8+O(a^6g^3)\ldots$$

but the application of (20) in [2] is not confined to small g and a. However, comparison of energies derived from the versions (20) and (22) indicates that the change, while in the correct direction, is not substantial. For example, taking $g=k=2$, $n=0$, which is the case (among those presented in [2]) which gives the largest departure from exact values, the energies are 1.485h from (20), 1.450h from (22) and 1.332h given in [19].

5. Energy equation for cubic and fifth-power oscillators

For the trial solution $x = a\cos(pt)$ the standard harmonic balance treatment of the cubic oscillator (1) (acceleration equation) gives [17] $p^2 = 3a^2k/4$. Eliminating harmonics from the energy equation gives $p^2 = a^2k/2$. This is a much worse estimate; the exact solution using an elliptic function gives [17] $p^2/a^2k = 0.7178$. Here, writing the exact solution as $Acn(t)$, with period T, we interpret a as A and p as $2\pi/T$.

To bring out the increasing complexity of the method with increasing order of approximation, we present the equations in first, second and third orders for the cubic oscillator. Using acceleration, these are

$$3a^2k - 4p^2 = 0 \tag{23}$$

$$\begin{aligned} &k(3a^2 + 3ab + 6b^2) - 4p^2 = 0 \\ &k(a^3 + 6a^2b + 3b^3) - 36bp^2 = 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} &k(3a^3 + 3a^2b + 6a(b^2 + bd + d^2) + 3b^2d) - 4ap^2 = 0 \\ &k(a^3 + 3a^2(2b + d) + 6abd + 3b(b^2 + 2d^2)) - 36bp^2 = 0 \\ &k(3a^2(b + 2d) + 3ab^2 + 3d(2b^2 + d^2)) - 100dp^2 = 0 \end{aligned} \tag{25}$$

while using energy they are

$$a^2k - 2p^2 = 0 \tag{26}$$

$$\begin{aligned} &k(a^3 + 3a^2b + 3ab^2 + 3b^3) + 12bp^2 - 2ap^2 = 0 \\ &k(a^3 + 12a^2b + 6ab^2 + 12b^3) - 48bp^2 = 0 \end{aligned} \tag{27}$$

and

$$\begin{aligned} &k(a^4 + a^3(3b + d) + 3a^2(b + d)^2 + 3ab(b^2 + bd + 2d^2) + 3bd(b^2 + d^2)) \\ &\qquad + 2p^2(-a^2 + 6ab + 30bd) = 0 \\ &k(a^4 + 12a^3(b + d) + 6a^2b(b + 2d) + 12a(b^3 + 2b^2d + 2bd^2 + d^3) \\ &\qquad + 6bd(b + d)) + 4ap^2(-12b + 20d) = 0 \\ &k(a^3(b + 3d) + 3a^2b(b + d) + 3ad(2b^2 + bd + d^2) \\ &\qquad + b^2(b^2 + 3d^2)) - p^2(20ad + 18b^2) = 0. \end{aligned} \tag{28}$$

We now go dierctly to the numerical solutions. In table 2 we present values for the cubic and fifth-power oscillators (1) and (5). These show how the two methods converge as the order increases, more rapidly for the cubic oscillator than for the fifth-power one. For the cubic oscillator the exact solution for the frequency [17] has now to be interpreted as

$$p^2/a^2k = 0.7178(1 + b/a + d/a)^2$$

where the quantity in brackets is needed so that the initial amplitude, $a + b + d$, can be identified with A. Using the third-order estimates of b/a and d/a, we have $p^2/a^2 = 0.7868$, so that the third-order result for p^2/a^2k is an excellent approximation. In the case of the fifth-power oscillator, with the most rapidly converging method of those

Table 2. Comparison of the results found using the acceleration equation and the energy equation, for cubic and fifth-power oscillators.

		p^2/a^2k	b/a	d/a
Cubic oscillator				
Acceleration	1st order	0.75	—	—
	2nd	0.7866	0.0448	—
	3rd	0.7869	0.0451	0.0019
Energy	1st order	0.5	—	—
	2nd	0.7543	0.0417	—
	3rd	0.7841	0.0448	0.0018
		p^2/a^4k	b/a	d/a
Fifth-power oscillator				
Acceleration	1st order	0.625	—	—
	2nd	0.7462	0.0665	—
	3rd	0.7566	0.0683	0.0093
Energy	1st order	0.3125	—	—
	2nd	0.6003	0.0559	—
	3rd	0.7154	0.0656	0.0076

examined in [6], p^2 lies above $0.9a^4k$. So in this case neither of our calculations is good, with the energy one somewhat worse than the acceleration one.

In these calculations we use the root-finding package *c05bnf* from the NAg library. This requires as input the algebraic equations, obtained in FORTRAN form from a REDUCE program, and initial estimates of the solutions, for which we use $p/a=1$, $b/a=0.1$, $d/a=0.01$.

6. Conclusions

Working at first order, we have shown that the use of integration to extract the appropriate Fourier component from a nonlinear damping term in two equations of the van der Pol type, with a denominator in the nonlinear function, is preferable to the use of a preliminary rationalization. The limit cycle amplitude obtained is consistent with bifurcation theory in both cases, but fits numerical simulations better if integration is used, as well as matching the amplitude obtained by other techniques. In this problem we have obtained second-order corrections (for the pre-rationalized method) and examined how they depend on the nonlinearity parameter q and the bifurcation parameter $|g-1|$. For another oscillator of this general type, we again find different results for the amplitude using the two methods, away from the immediate vicinity of the bifurcation. Again the results of the integration method are in better agreement with other calculations.

Comparing results of pre-rationalization and integration in a simple conservative oscillator used to illustrate harmonic balance as a tool in quasi-classical quantization, the energies calculated are altered in the correct direction when integration is used, but not substantially.

At first order the technique of eliminating the first-harmonic term from the energy gives a poor estimate of the frequency of a cubic oscillator. At second and third order the technique gives results that converge towards those of the standard harmonic balance method. Results for a fifth-power oscillator are not so satisfactory.

Acknowledgments

Some of the results presented originated in student projects on computer algebra applications. Mr A Wilson obtained the results for cubic and fifth-power oscillators using REDUCE 3.3. I wish to thank Dr A Watt for making available an o.d.e. simulation program.

Appendix 1. The two methods compared from the viewpoint of minimizing energy fluctuations

We wish to compare two harmonic balance second-order methods from the viewpoint that the one giving lower fluctuations in the total energy is to be preferred, since any fluctuation in the energy is an artefact of the approximation. Note that here we include all harmonics present in the energy, not just the $\cos(2pt)$ part. We study the linear + cubic oscillator

$$\mathrm{d}^2x/\mathrm{d}t^2 + kx + mx^3 = 0 \tag{29}$$

with energy

$$(\mathrm{d}x/\mathrm{d}t)^2/2 + kx^2/2 + mx^4/4. \tag{30}$$

We use the trial solutions

$$x_1 = \cos(pt) + b\cos(3pt) \tag{31}$$

$$x_2 = \cos(pt)/(1 + c\cos(2pt)). \tag{32}$$

Using these we have approximately

$$\begin{aligned} p^2 &= (k + 11m/14) \\ b &= 1/(21 + 32k/m) \end{aligned} \tag{33}$$

and

$$\begin{aligned} p^2 &= \frac{192k^2 + 281km + 100m^2}{3(64k + 43m)} \\ c &= -1/(10 + 16k/m) \end{aligned} \tag{34}$$

respectively. We examine a region of (k, m) space in which these values lead to fluctuations of a few per cent in the energy (30). The results in Table 3 confirm that the version (32)—rational trial solution—can give substantially lower fluctuations in the energy.

Table 3. Comparison of the fluctuations in the energy (30) of the anharmonic oscillator (29) using the trial solutions (31) and (32), for a number of values of k and m. In each pair the upper number is $E_{max} - E_{min}$ for the solution (31), the lower is $E_{max} - E_{min}$ for the solution (32).

k	m	3/4	1	5/4
3/4		75	98	119
		41	105	197
1		80	103	123
		5	55	122
5/4		85	87	128
		20	18	71

It thus appears that in any application of harmonic balance to conservative systems it is wise, if it is practical, to compare standard and rational trial solutions in this way. The rational version requires a more elaborate calculation.

Appendix 2. Averaging compared with harmonic balance

Consider the equations

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= y \\ \mathrm{d}y/\mathrm{d}t + x + qF(x, y) &= 0 \end{aligned} \tag{35}$$

for which Hayashi [20] gives a good description of the averaging method. Harmonic balance, at lowest order, assumes a solution $x = a\cos(pt)$, implying $y = -ap\sin(pt)$, and obtains a and p as we have described. The averaging method assumes a solution $x = a(t)\cos(t)$, $y = b(t)\sin(t)$ and obtains a pair of integrals for $\mathrm{d}a/\mathrm{d}t$ and $\mathrm{d}b/\mathrm{d}t$, or alternatively for $\mathrm{d}r/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ in a polar coordinate version. The amplitude given by this method is the value r_0 for which $\mathrm{d}r/\mathrm{d}t = 0$. A non-zero value of $\mathrm{d}\theta/\mathrm{d}t$, using r_0, can be interpreted as a correction to the estimate $p = 1$.

Suppose now that we adopt the particular form

$$F(x, \mathrm{d}x/\mathrm{d}t) = f(x^2)\,\mathrm{d}x/\mathrm{d}t \tag{36}$$

as in all the dissipative oscillators treated in this paper, and use the integration version of first-order harmonic balance. The integral $I(r)$ representing $\mathrm{d}r/\mathrm{d}t$ in the averaging method has the form $I(a, 1)$, where $pI(a, p)$ gives the coefficient of $\sin(pt)$ in the harmonic balance method. Since in this special case $p = 1$, the amplitude a is the same as the amplitude r_0.

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